Vector Calculus: Grad, Div and Curl

In vector calculus, div, grad and curl are standard differentiation[1] operations on scalar or vector fields, resulting in a scalar or vector[2] field.

Scalar and Vector fields

A scalar field is one that has a single value associated with each point in the domain. A simple example is a temperature distribution; every point in the domain has a single value.

A vector field has a magnitude and direction associated with each point in the domain. An example is a velocity field in a fluid; every point in the field has a velocity vector associated with it. If the vector is resolved into components then each point in a two dimensional vector field has two values associated with it and each point in a three-dimensional vector field has three values associated with it.

The definitions also bring in one new symbol of notation;' ∇' – called 'del' or 'nabla'. The del operator can be defined as:

$$\nabla \equiv \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix}$$

for two-dimensional problems and

$$\nabla \equiv \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix}$$

for three-dimensional problems. The del operator can be applied directly to a scalar field (grad), or it may be applied to a vector field through either dot product (div) or cross product (curl)[3].

Grad: The gradient of a scalar field

The gradient of a scalar field is a vector field and whose magnitude is the rate of change and which points in the direction of the greatest rate of increase of the scalar field. If the vector is resolved, its components represent the rate of change

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[1] Partial Differentiation
[2] Vectors and Scalars
[3] Vector Arithmetic

of the scalar field with respect to each directional component. Hence for a two-dimensional scalar field $\varphi(x, y)$,

$$\text{grad } \varphi(x,y) = \nabla\varphi(x,y) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \varphi = \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \end{pmatrix},$$

and for a three-dimensional scalar field $\varphi(x, y, z)$,

$$\text{grad } \varphi(x,y,z) = \nabla\varphi(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \varphi = \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \\ \frac{\partial \varphi}{\partial z} \end{pmatrix},$$

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Note that the gradient of a scalar field is a vector field.

Example 1 For the scalar field

$$\varphi(x,y) = 3x + 2y\,,$$

$$\nabla\varphi(x,y) = \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{or } 3\,\mathbf{i} * 2\,\mathbf{j}.$$

Example 2 For the scalar field

$$\varphi(x,y,z) = x^2yz,$$

$$\nabla\varphi(x,y,z) = \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \\ \frac{\partial \varphi}{\partial z} \end{pmatrix} = \begin{pmatrix} 2xyz \\ x^2z \\ x^2y \end{pmatrix}.$$

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Div: the divergence of a vector field

For a two-dimensional vector field $\boldsymbol{f}(x,y) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$,

$$\text{div}\,\boldsymbol{f} = \nabla.\boldsymbol{f}(x,y) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}.\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}.$$

For a three-dimensional vector field $\boldsymbol{f}(x,y,z) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$,

$$\text{div}\,\boldsymbol{f} = \nabla.\boldsymbol{f}(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}.\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

<u>Example 3</u>

For the vector field $\boldsymbol{f}(x,y,z) = \begin{pmatrix} x+y \\ xy \end{pmatrix}$,

$$\nabla.\boldsymbol{f}(x,y,z) = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(xy) = 1 + x$$

<u>Example 4</u>

For the vector field $\boldsymbol{f}(x,y,z) = \begin{pmatrix} x^2y \\ x^2z \\ y^2xz \end{pmatrix}$,

$$\nabla.\boldsymbol{f}(x,y,z) = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(x^2z) + \frac{\partial}{\partial z}(y^2xz) = 2xy + 0 + y^2x$$

Curl

The curl vector is the normal around which the greatest circulation exists for that point in the vector field. The curl for the three-dimensional vector field $\boldsymbol{f}(x, y, z) = (f_x, f_y, f_z)$ is defined as follows:

$$\nabla \times \boldsymbol{f} = \nabla \times \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{pmatrix} \dfrac{\partial f_3}{\partial y} - \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_1}{\partial z} - \dfrac{\partial f_3}{\partial x} \\ \dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial y} \end{pmatrix} \text{ or } \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}.$$

<u>Example 5</u>

For the vector field $\boldsymbol{f}(x, y, z) = \begin{pmatrix} x^2 y \\ x^2 z \\ y^2 xz \end{pmatrix}$,

$$\nabla \times \boldsymbol{f}(x, y, z) = \begin{pmatrix} 2xyz - x^2 \\ y^2 z \\ 2xz - x^2 \end{pmatrix} = (2xyz - x^2)\mathbf{i} - (y^2 z)\mathbf{j} + (2xz - x^2)\mathbf{k}$$